

# THE GOLOD PROPERTY FOR PRODUCTS AND HIGH SYMBOLIC POWERS OF MONOMIAL IDEALS

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**ABSTRACT.** We show that for any two proper monomial ideals  $I$  and  $J$  in the polynomial ring  $S = k[x_1, \dots, x_n]$  the ring  $S/IJ$  is Golod. We also show that if  $I$  is squarefree then for large enough  $k$  the quotient  $S/I^{(k)}$  of  $S$  by the  $k^{\text{th}}$  symbolic power of  $I$  is Golod. As an application we prove that the multiplication on the cohomology algebra of some classes of moment-angle complexes is trivial.

## 1. INTRODUCTION

For a graded ideal  $I$  in the polynomial ring  $S = \mathbb{K}[x_1, \dots, x_n]$  in  $n$  variables over the field  $\mathbb{K}$  the ring  $S/I$  is called *Golod* if all Massey operations on the Koszul complex of  $S/I$  with respect of  $\mathbf{x} = x_1, \dots, x_n$  vanish. The naming gives credit to Golod [11] who showed that the vanishing of the Massey operations is equivalent to the equality case in the following coefficientwise inequality of power-series which was first derived by Serre:

$$\sum_{i \geq 0} \dim_{\mathbb{K}} \operatorname{Tor}_i^{S/I}(\mathbb{K}, \mathbb{K}) t^i \leq \frac{(1+t)^n}{1 - t \sum_{i \geq 1} \dim_{\mathbb{K}} \operatorname{Tor}_i^S(S/I, \mathbb{K}) t^i}$$

We refer the reader to [1] and [8] for further information on the Golod property and to [4] and [12] for the basic concepts from commutative algebra underlying this paper. We prove the following two results.

**Theorem 1.1.** *Let  $I, J$  be two monomial ideals in  $S$  different from  $S$ . Then  $S/IJ$  is Golod.*

In the statement of the second result we write  $I^{(k)}$  for the  $k^{\text{th}}$  symbolic power of the ideal  $I$ .

**Theorem 1.2.** *Let  $I$  be a squarefree monomial ideal in  $S$  different from  $S$ . Then for  $k \gg 0$  the  $k^{\text{th}}$  symbolic power  $I^{(k)}$  is Golod for  $k \gg 0$ .*

Besides the strong algebraic implications of Golodness the case of squarefree monomial ideals relates to interesting topology. Let  $\Delta$  be a simplicial complex on ground set  $[n]$  and let  $\mathbb{K}[\Delta]$  be its Stanley–Reisner ring (see § 4 for basic facts about Stanley–Reisner rings).

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By work of Buchstaber and Panov [5, Thm. 7.7], extending an additive isomorphism from [10], it is known that there is an algebra isomorphism of the Koszul homology  $H_*(\mathbf{x}, k[\Delta])$  and the singular cohomology ring  $H^*(M_\Delta; k)$  where  $M_\Delta = \{(v_1, \dots, v_n) \in (D^2)^n \mid \{i \mid v_i \notin S^1\} \in \Delta\}$ . Here  $D^2 = \{v \in \mathbb{R}^2 \mid \|v\| \leq 1\}$  is the unit disk in  $\mathbb{R}^2$  and  $S^1$  its bounding unit circle. Note that the isomorphism is not graded for the usual grading of  $H_*(\mathbf{x}, k[\Delta])$  and  $H^*(M_\Delta; k)$ . The complex  $M_\Delta$  is the *moment-angle complex* or *polyhedral product* of the pair  $(D^2, S^1)$  for  $\Delta$  (we refer the reader to [5] and [7] for background information). Last we write  $\Delta^\circ = \{A \subseteq [n] \mid [n] \setminus A \notin \Delta\}$  for the *Alexander dual* of the simplicial complex  $\Delta$ . Now we are in position to formulate the following consequence of Theorem 1.1.

**Corollary 1.3.** *Let  $\Delta$  be a simplicial complex such that  $\Delta = (\Delta_1^\circ * \Delta_2^\circ)^\circ$  for two simplicial complexes  $\Delta_1, \Delta_2$  on disjoint ground sets. Then the multiplication on  $H^*(M_\Delta; k)$  is trivial.*

The main tool for the proof of Theorem 1.1 and Theorem 1.2 is combinatorial. Let  $I$  be a monomial ideal and write  $G(I)$  for the set of minimal monomial generators of  $I$ . In [16, Def. 3.8] the author introduces a combinatorial condition on  $G(I)$  that in [3] was shown to imply the Golod property for  $S/I$ . The ideal  $I$  is said to satisfy the *strong gcd-condition* if there exists a linear order  $\prec$  on  $G(I)$  such that for any two monomials  $u \prec v \in G(I)$  with  $\gcd(u, v) = 1$  there exists a monomial  $w \in G(I)$  with  $w \neq u, v$  such that  $u \prec w$  and  $w$  divides  $\text{lcm}(u, v) = uv$ . The following result from [3] removes an unnecessary assumption from the statement of Theorem 7.5 in [16].

**Theorem 1.4** (Thm. 5.5 [3]). *Let  $I$  be a monomial ideal. If  $I$  satisfies the strong gcd-condition then  $S/I$  is Golod.*

We refer the reader to [2] for the relation of the gcd-condition to standard combinatorial properties of simplicial complexes.

The paper is organized as follows. In § 2 we verify the strong gcd-condition for products of monomial ideals and in § 3 for high symbolic powers of squarefree monomial ideals. This yields Theorem 1.1 and Theorem 1.2. In § 4 we study the implications of Theorem 1.1 and Theorem 1.2 on moment-angle complexes. In particular, we derive Corollary 1.3.

## 2. PRODUCT OF MONOMIAL IDEALS

Since by Theorem 1.4 for any monomial ideal  $I$  satisfying the strong gcd-condition the ring  $S/I$  is Golod, the following result immediately implies Theorem 1.1.

**Proposition 2.1.** *For any two monomial ideals  $I, J$  in  $S$  that are different from  $S$  the ideal  $IJ$  satisfies strong-gcd condition.*

*Proof.* Fix a monomial order  $<$  on the set of monomials of  $S$ . We define a linear order  $\prec$  on  $G(IJ)$  as follows: For two monomials  $u, v \in G(IJ)$ , we set  $u \prec v$  if and only if  $\deg(u) > \deg(v)$  or  $\deg(u) = \deg(v)$  and  $u < v$ . Now the following claim states that  $G(IJ)$  ordered by  $\prec$  satisfies the strong gcd-condition.

**Claim:** For any two monomials  $u, v \in G(IJ)$  with  $u \prec v$  and  $\gcd(u, v) = 1$  there exists a monomial  $w \in G(IJ)$  different from  $u$  and  $v$  such that  $u \prec w$  and  $w$  divides  $\text{lcm}(u, v) = uv$ .

Since  $u, v \in G(IJ)$ , there exist monomials  $u_1, v_1 \in G(I)$  and  $u_2, v_2 \in G(J)$  such that  $u = u_1u_2$  and  $v = v_1v_2$ . We consider two cases:

**Case 1.**  $\deg(u) = \deg(v)$ .

By definition,  $u < v$ . Therefore,  $u_1 < v_1$  or  $u_2 < v_2$ , since otherwise  $u = u_1u_2 \geq v_1v_2 = v$ , which is a contradiction. Without loss of generality we may assume that  $u_1 < v_1$ .

If  $\deg(u_1) \geq \deg(v_1)$ , then define  $w' := v_1u_2$ . Now  $w' \in IJ$ ,  $\deg(w') \leq \deg(u)$  and  $u < w'$ . If  $w' \in G(IJ)$ , then we set  $w = w'$  and clearly  $u < w$  and  $w$  divides  $\text{lcm}(u, v) = uv$ . By  $\gcd(u, v) = 1$ , it follows that  $u_2 \neq v_2$  and therefore  $w \neq v$ . If  $w' \notin G(IJ)$ , then there exists a monomial  $w \in G(IJ)$ , such that  $w|w'$ . Now  $w$  divides  $\text{lcm}(u, v) = uv$  and since  $\deg(w) < \deg(w') \leq \deg(u) = \deg(v)$ , we conclude that  $u < v < w$  and  $w \neq v$ .

If  $\deg(u_1) < \deg(v_1)$ , then  $\deg(u_2) > \deg(v_2)$ , since  $\deg(u) = \deg(v)$ . We define  $w' := u_1v_2$ . Now  $w' \in IJ$  and therefore there exists a monomial, say  $w \in G(IJ)$ , such that  $w|w'$ . Then  $w$  divides  $\text{lcm}(u, v) = uv$  and since  $\deg(w) \leq \deg(w') < \deg(u) = \deg(v)$ , we conclude that  $u < v < w$  and  $w \neq v$ .

**Case 2.**  $\deg(u) \neq \deg(v)$ .

By definition of  $<$  we must have  $\deg(u) > \deg(v)$ . Hence either of  $\deg(v_1) < \deg(u_1)$  or  $\deg(v_2) < \deg(u_2)$  holds. Without loss of generality we assume that  $\deg(v_1) < \deg(u_1)$ . We define  $w' = v_1u_2$ . Then  $w' \in IJ$  and therefore there exists a monomial, say  $w \in G(IJ)$ , such that  $w$  divides  $w'$ . Thus  $w$  divides  $\text{lcm}(u, v) = uv$  and  $u < w$ , since  $\deg(w) \leq \deg(w') < \deg(u)$ . Also  $w \neq v$ , because otherwise  $v$  divides  $v_1u_2$  and since  $\gcd(u_1, v) = 1$ , it implies that  $v$  divides  $v_1$ , which is impossible.  $\square$

In [15, Thm. 4.1] it is shown that for a homogeneous ideal  $I \neq S$  in the polynomial ring  $S$  the ring  $S/I^k$  is Golod for  $k \gg 0$ . For monomial ideals Theorem 1.1 allows a more precise statement of their result which follows immediately from Proposition 2.1 and Theorem 1.4.

**Corollary 2.2.** *Let  $I$  be a monomial ideal in the polynomial ring  $S$  different from  $S$ . Then for  $k \geq 2$  the ideal  $I^k$  satisfies the strong gcd-condition and hence the ring  $R = S/I^k$  is Golod.*

Since by [4, Thm. 3.4.5] the Koszul homology  $H_*(\mathbf{x}, S/I)$  for a Gorenstein quotient  $S/I$  of  $S$  is a Poincaré duality algebra,  $S/I$  cannot be Golod unless  $I$  is a principal ideal. Thus Theorem 1.1 implies that a product of monomial ideal can be Gorenstein if and only if it is principal. This observation is not new and can be seen as a consequence of a very general result by Huneke [14] who showed that in any unramified regular local ring no Gorenstein ideal of height  $\geq 2$  can be a product.

### 3. SYMBOLIC POWERS OF SQUAREFREE MONOMIAL IDEALS

For our results on symbolic powers we have to restrict ourselves to squarefree monomial ideals  $I$ . This is due to the fact that our proofs use that only in this case in the primary decomposition  $I = \mathfrak{p}_1 \cap \dots \cap \mathfrak{p}_r$  of  $I$  every  $\mathfrak{p}_i$  is an ideal of  $S$  generated by a subset of the variables of  $S$  [12, Lem. 1.5.4]. Moreover, in this situation for a positive integer  $k$  the  $k^{\text{th}}$  symbolic power  $I^{(k)}$  of  $I$  coincides with  $\mathfrak{p}_1^k \cap \dots \cap \mathfrak{p}_r^k$  [12, Prop.1.4.4].

Now we are ready to prove that the high symbolic powers of squarefree monomial ideals of  $S$  fulfill the strong-gcd condition.

**Proposition 3.1.** *Let  $I$  be a squarefree monomial ideal in  $S$  different from  $S$ . Then for  $k \gg 0$  the  $k^{\text{th}}$  symbolic power  $I^{(k)}$  satisfies the strong gcd-condition.*

*Proof.* By [17, Prop. 1] the ring  $A = \bigoplus_{i=0}^{\infty} I^{(i)}$  is Noetherian and therefore is a finitely generated  $\mathbb{K}$ -algebra. Assume that the set  $\{y_1, \dots, y_m\}$  is a set of generators for the  $\mathbb{K}$ -algebra  $A$ . Following [6], since  $A$  is finitely generated, there exists a natural number  $c$  such that  $A_0 = \bigoplus_{i=0}^{\infty} I^{(ci)}$  is a standard  $\mathbb{K}$ -algebra and therefore a Noetherian ring. Note that the set

$$\{y_1^{\ell_1} \dots y_m^{\ell_m} \mid 0 \leq \ell_1, \dots, \ell_m \leq c-1\}$$

is a system of generators for  $A$  as an  $A_0$ -module. Assume that the degree of a generator of the  $A_0$ -module  $A$  is at most  $\alpha$ . Since  $A_0$  is standard  $\mathbb{K}$ -algebra for every integer  $k > \max\{\alpha, c\}$  we have  $I^{(k)} = I^{(c)}I^{(k-c)}$ . Thus for every  $k > \max\{\alpha, c\}$ , the ideal  $I^{(k)}$  is the product of two monomial ideals. Therefore, by Proposition 2.1 it satisfies the strong-gcd condition.  $\square$

The following corollary is an immediate consequence of Theorem 1.4 and Theorem 1.2.

**Corollary 3.2.** *Let  $I$  be a squarefree monomial ideal in the polynomial ring  $S$ , which is not a principal ideal. Then for  $k \gg 0$  the ring  $S/I^{(k)}$  is not Gorenstein.*

We do not know which  $k$  suffices. Indeed, we do not have an example of a monomial ideal  $I \neq S$  squarefree or not for which  $I^{(2)}$  is not Golod.

#### 4. MOMENT-ANGLE COMPLEXES

First, we recall some basics from the theory of Stanley-Reisner ideals. Let  $\Delta$  be a simplicial complex on ground set  $[n]$ . A subset  $N \subseteq [n]$  such that  $N \notin \Delta$  and  $N \setminus \{i\} \in \Delta$  for all  $i \in N$  is called a *minimal non-face* of  $\Delta$ . The *Stanley-Reisner ideal*  $I_{\Delta}$  of  $\Delta$  is the ideal in  $S$  generated by the monomial  $x_N$  for the minimal non-faces  $N$  of  $\Delta$  and the quotient  $k[\Delta] = S/I_{\Delta}$  is Stanley-Reisner ring of  $\Delta$ . Indeed, the map sending  $\Delta$  to  $I_{\Delta}$  is a bijection between the simplicial complexes on ground set  $[n]$  and the squarefree monomial ideals in  $S$ . To any monomial ideal  $I$  its *polarization*  $I^{\text{pol}}$  [12, p. 19] is a squarefree monomial ideal and it is known [9, Thm. 3.5] that  $I$  is Golod if and only if  $I^{\text{pol}}$  is Golod. In addition, it follows from the vanishing of the Massey operations that the multiplication on Koszul homology  $H_*(\mathbf{x}, S/I)$  is trivial for all Golod  $S/I$  – by trivial we here mean that products of two elements of positive degree are 0. Thus the algebra isomorphism of  $H_*(\mathbf{x}, k[\Delta])$  and  $H^*(M_{\Delta}; k)$  [5, Thm. 7.7] together with Theorem 1.1 and Theorem 1.2 yields the following corollary. Note that even though the isomorphism of  $H_*(\mathbf{x}, k[\Delta])$  and  $H^*(M_{\Delta}; k)$  is not graded in the usual grading it sends  $H_0(\mathbf{x}, k[\Delta])$  and  $H^0(M_{\Delta}; k)$ .

**Corollary 4.1.** *Let  $I$  and  $J$  be monomial ideals. Then for the simplicial complexes  $\Gamma$  and  $\Gamma^{(k)}$  such that  $(IJ)^{\text{pol}} = I_{\Gamma}$  and  $(I^{(k)})^{\text{pol}} = I_{\Gamma^{(k)}}$  we have:*

- (i) *The multiplication on the cohomology algebra of  $M_{\Gamma}$  is trivial.*

(ii) *The multiplication on the cohomology algebra of  $M_{\Gamma^{(k)}}$  is trivial for  $k \gg 0$ .*

In general, the combinatorics and geometry of the simplicial complexes  $\Gamma$  and  $\Gamma^{(k)}$  cannot be easily controlled even in the case  $I$  and  $J$  are squarefree monomial ideals. Therefore, the result is more useful in a situation when the ideals  $IJ$  and  $I^{(k)}$  themselves are squarefree monomial ideals. Since this never happens for  $I^{(k)}$  and  $k \geq 2$  we confine ourselves to the case of products  $IJ$ . The following lemma shows that in this case  $I$  and  $J$  should be squarefree monomial ideals with generators in disjoint sets of variables. Even though the lemma must be a known basic fact from the theory of the monomial ideals we did not find a reference and hence for the sake of completeness we provide a proof. In the proof we denote for a monomial  $u$  by  $\text{supp}(u)$  its *support*, which is the set of variables dividing  $u$ .

**Lemma 4.2.** *Let  $I, J$  be monomial ideals. Then  $IJ$  is a squarefree monomial ideal if and only if  $I$  and  $J$  are squarefree monomial ideals such that  $\gcd(u, v) = 1$  for all  $u \in G(I)$  and  $v \in G(J)$ .*

*Proof.* The “if” part of the lemma is trivial. The other direction states that if  $IJ$  is a squarefree monomial ideal then for every  $u \in G(I)$  and every  $v \in G(J)$  the monomial  $uv$  is squarefree. Assume by contradiction that there exist monomials  $u \in G(I)$  and  $v \in G(J)$  such that  $uv$  is not squarefree. Among every  $v$  with this property we choose one such that the set  $\text{supp}(v) \setminus \text{supp}(u)$  has minimal cardinality. Since  $IJ$  is a squarefree monomial ideal, there exist squarefree monomials  $u' \in G(I)$  and  $v' \in G(J)$  such that  $\text{supp}(u') \cap \text{supp}(v') = \emptyset$  and  $u'v'$  divides  $uv$ . We distinguish two cases:

**Case:**  $\text{supp}(v) \setminus \text{supp}(u) = \emptyset$

The assumption is equivalent to  $\text{supp}(v) \subseteq \text{supp}(u)$ . We conclude that  $\text{supp}(u') \subseteq \text{supp}(uv) = \text{supp}(u)$ . But then  $u'$  divides  $u$  and hence  $u = u'$ . But then  $\text{supp}(uv) = \text{supp}(u')$  and  $v' = 1$ . This implies  $J = S$  and  $IJ = I$  which in turn shows that  $I$  is a squarefree monomial ideal. Thus  $u$  is squarefree and  $v = v' = 1$ . But then  $uv$  is squarefree and we arrive at a contradiction.

**Case:**  $\text{supp}(v) \setminus \text{supp}(u) = \{x_{i_1}, \dots, x_{i_t}\}$  for some  $t \geq 1$

If  $\text{supp}(v') \subseteq \text{supp}(v)$  then  $v'$  divides  $v$  and hence  $v = v'$ . Then  $u'$  must divide  $u$  and hence  $u = u'$ . But this contradicts the fact that  $uv = u'v'$  is not squarefree. Hence  $\text{supp}(u) \cap \text{supp}(v') \neq \emptyset$  and therefore  $uv'$  is not squarefree. Now assume that  $\{x_{i_1}, \dots, x_{i_t}\} \subseteq \text{supp}(v')$ . Then since  $\text{supp}(u') \cap \text{supp}(v') = \emptyset$ , it follows that  $\text{supp}(u') \subseteq \text{supp}(u)$  and therefore  $u'$  divides  $u$ . As above this yields a contradiction. Thus  $\{x_{i_1}, \dots, x_{i_t}\} \not\subseteq \text{supp}(v')$  and the inclusion  $\text{supp}(v') \subseteq \text{supp}(u) \cup \text{supp}(v)$  implies that the cardinality of  $\text{supp}(v') \setminus \text{supp}(u)$  is strictly less than the cardinality of  $\text{supp}(v) \setminus \text{supp}(u)$ , which contradicts the choice of  $v$ .  $\square$

Let us analyze the situation when  $IJ$  is a squarefree monomial ideal more carefully. By Lemma 4.2, we may assume that  $I = I_\Delta$  and  $J = I_{\Delta'}$  with simplicial complexes  $\Delta$  and  $\Delta'$  that have a join decomposition  $\Delta = 2^{V_1} * \Delta_1$  and  $\Delta' = 2^{V_2} * \Delta_2$  for simplicial complexes  $\Delta_1$  and  $\Delta_2$  over disjoint ground sets and full simplices  $2^{V_1}$  and  $2^{V_2}$  over arbitrary finite sets  $V_1$  and  $V_2$ . The product of the corresponding Stanley–Reisner ideals is easily described.

**Lemma 4.3.** *Let  $\Delta_i$ ,  $i = 1, 2$ , be simplicial complexes on disjoint ground sets  $\Omega_1$  and  $\Omega_2$ . Then  $I_{\Delta_1} I_{\Delta_2} = I_{(\Delta_1^\circ * \Delta_2^\circ)^\circ}$ .*

*Proof.* Recall that by definition of the Alexander dual, for every simplicial complex  $\Delta$  on the ground set  $\Omega$ , the maximal faces  $A \in \Delta^\circ$  are those subsets  $A$  of  $\Omega$  for which  $\Omega \setminus A$  is a minimal non-face of  $\Delta$ .

The product  $I_{\Delta_1} I_{\Delta_2}$  is generated by  $x_{N_1} x_{N_2}$  for minimal non-faces  $N_1$  and  $N_2$  of  $\Delta_1$  and  $\Delta_2$  respectively. Hence it is generated by monomials corresponding to the union of  $F_1 = \Omega_1 \setminus N_1$  and  $F_2 = \Omega_2 \setminus N_2$  of maximal faces of  $\Delta_1^\circ$  and  $\Delta_2^\circ$ . Since for any maximal face  $F$  of  $\Delta_1^\circ * \Delta_2^\circ$  there are maximal faces  $F_1$  of  $\Delta_1^\circ$  and  $F_2$  of  $\Delta_2^\circ$  such that  $F = F_1 \cup F_2$  it follows that the monomials  $x_{N_1 \cup N_2} = x_{N_1} x_{N_2}$  for minimal non-faces  $N_1$  of  $\Delta_1$  and  $N_2$  of  $\Delta_2$  are the generators  $I_{(\Delta_1^\circ * \Delta_2^\circ)^\circ}$ . Now the assertion follows.  $\square$

Now combining Lemma 4.3 and Corollary 4.1 (i) yields Corollary 1.3.

We note, that for the deduction of Corollary 1.3 one could have also argued using [13, Satz 2] instead of Theorem 1.1.

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